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Bose–Einstein condensation of a relativistic Bose gas trapped in a general external potential

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Abstract
Bose–Einstein condensation of an ideal relativistic Bose gas trapped in a generic power-law potential is investigated. The analytical expressions for some important parameters such as the critical temperature, ground-state fraction and heat capacity are derived. The general criteria on the occurrence of Bose–Einstein condensation and the discontinuity of heat capacity at the critical temperature are obtained. The results obtained here present a unified description for the Bose–Einstein condensation of a class of ideal Bose systems so that many important conclusions in the literature are included in this paper.

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1. Introduction

It is commonly expected that the relativistic corrections are negligible near the critical temperature of Bose–Einstein condensation (BEC), since the kinetic energy of a boson at $T \sim T_C$ is generally much smaller than its static energy, i.e., $k_B T \ll mc^2$, where m is the rest mass of a boson, c is the velocity of light and k_B is the Boltzmann constant. The expectation, however, is not always true. It is found that some systems in the universe may consist of bosons of very small rest mass. For example, the boson of a pair of neutrinos is of the rest mass about 10^{-30} g [1]. For the systems of such bosons, the condition $k_B T \ll mc^2$ is often not satisfied, and thus the relativistic effects may be considerable.

The properties of relativistic Bose gases have been investigated by several authors [1–7]. Early in 1965, Landsberg and Dunning-Davies studied the condensation temperature and the anomaly of the heat capacity of an ideal relativistic Bose gas [1]. Beckmann *et al* extended the work and discussed the relativistic BEC in various spatial dimensions [2, 3], particular attentions being focused on the dependence of BEC on the dimensionality. A more complete treatment of BEC for the relativistic systems was given by using the techniques of quantum-field theory at the finite temperature and density, in which the possibility of particle–antiparticle pair production was taken into account [4–7].

Most of the previous investigations on the characteristics of relativistic Bose gases, to our knowledge, are mainly concentrated on the systems in the absence of external potential. It has been shown that the constrained role of the external potential may significantly change the performance of Bose gases [8–10]. It can be said that the external potential creates favourable conditions for controlling degenerate Bose gases and quantitatively investigating their performance. Therefore, it is necessary to give further investigations on the properties of relativistic Bose gases in the presence of the external potential.

In this paper, we study BEC characteristics of a relativistic Bose gas trapped in a generic power-law potential. Some important parameters are derived analytically and the general criteria on BEC occurrence and the discontinuity of the heat capacity at the critical temperature are obtained. The results obtained here provide a unified description of the low-temperature behaviours for a class of ideal Bose gases, from which many results in the literature can be derived in particular situations.

2. Theoretical evaluation

We consider an ideal relativistic Bose gas trapped in a D-dimensional conservative field. The single-particle Hamiltonian can be expressed as

$$H(\boldsymbol{p}, \boldsymbol{r}) = \sqrt{p^2c^2 + m^2c^4} + U(\boldsymbol{r}), \tag{1}$$

where $\boldsymbol{p}$ and $\boldsymbol{r}$ are, respectively, the momentum and coordinate of a particle, $U(\boldsymbol{r})$ is the potential corresponding to the conservative field. It can be proved that the Hamiltonian given above along with the momentum $\boldsymbol{p}$ constitutes a covariant vector $(\boldsymbol{p}, \mathrm{i}\varepsilon/c)$.

When the particle number in the system is large and the level spacing is much smaller than the mean kinetic energy of particles (this condition is often satisfied [10]), the Thomas-Fermi's semiclassical approximation is valid. Thus, the total number of particles N and the total energy E of the system can be expressed, respectively, as

$$N = N_0 + \frac{1}{h^D}\int \frac{\mathrm{d}^D\boldsymbol{p}\,\mathrm{d}^D\boldsymbol{r}}{\exp[\beta(\sqrt{p^2c^2 + m^2c^4} + U(\boldsymbol{r}) - \mu)] - 1}, \tag{2}$$

$$E = N_0mc^2 + \frac{1}{h^D}\int \frac{\sqrt{p^2c^2 + m^2c^4} + U(\boldsymbol{r})}{\exp[\beta(\sqrt{p^2c^2 + m^2c^4} + U(\boldsymbol{r}) - \mu)] - 1}\,\mathrm{d}^D\boldsymbol{p}\,\mathrm{d}^D\boldsymbol{r}, \tag{3}$$

where $\beta = 1/(k_\mathrm{B}T)$, h is the Planck constant, and

$$N_0 = \frac{1}{\exp[\beta(mc^2 - \mu)] - 1} \tag{4}$$

is the number of particles in the ground state.

We assume the external potential to be of the generic power-law form as

$$U(\boldsymbol{r}) = \sum_{k=1}^{D} \varepsilon_k \left|\frac{x_k}{L_k}\right|^{t_k}, \tag{5}$$

where x_k is the kth component of coordinate of a particle, t_k, ε_k and L_k are all positive constants that mark the shape and strength of the external potential. Substituting equation (5) into equations (2) and (3), one can get the expressions for the total number of particles and total energy as

$$N = N_0 + \frac{\tilde{V}}{\lambda^D}\left(\frac{2}{\pi\chi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta-1}}K_{D'}(j/\chi), \tag{6}$$

$$E = Nmc^2 + \frac{\tilde{V}k_BT}{\lambda^D}\left(\frac{2}{\pi\chi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta}}[(\eta-1-j/\chi)K_{D'}(j/\chi)+(j/\chi)K_{D'+1}(j/\chi)], \tag{7}$$

respectively,where $D' = (D+1)/2$, $\eta = \sum_{k=1}^{D} 1/t_k$, $\chi = k_BT/(mc^2)$, $\lambda = \sqrt{h^2/(2\pi mk_BT)}$ is the nonrelativistic thermal wavelength,

$$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{x}{2}\right)^\nu\int_0^\infty \exp(-x\cosh\theta)\sinh^{2\nu}\theta\,\mathrm{d}\theta \tag{8}$$

is the modified Bessel function, and

$$\tilde{V} = \prod_{k=1}^{D}\frac{(2L_k)\Gamma(1/t_k+1)}{(\beta\varepsilon_k)^{1/t_k}} \tag{9}$$

may be referred to as the effective volume of the system [11].

When $\mu \to mc^2$ and the number of particles in the ground state is still macroscopically negligible, i.e., $N_0 \to 0$, BEC begins to occur in the system. Thus, according to equation (6), the critical temperature T_C of BEC is determined by

$$N = \frac{\tilde{V}_C}{\lambda_C^D}\left(\frac{2}{\pi\chi_C}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}K_{D'}(j/\chi_C), \tag{10}$$

where

$$\lambda_C = \sqrt{h^2/(2\pi mk_BT_C)}, \qquad \chi_C = k_BT_C/(mc^2),$$

and

$$\tilde{V}_C = \prod_{k=1}^{D}\frac{(2L_k)\Gamma(1/t_k+1)}{(\beta_C\varepsilon_k)^{1/t_k}}. \tag{11}$$

From equations (6) and (10), the ground-state fraction at $T \leqslant T_C$ is found to be

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{D'+\eta-1}\frac{\sum_{j=1}^{\infty}\frac{\exp(j/\chi)}{j^{D'+\eta-1}}K_{D'}(j/\chi)}{\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}K_{D'}(j/\chi_C)}. \tag{12}$$

According to $C = (\partial E/\partial T)_N$, one can calculate the heat capacity at the given number of particles and external potential. When $T > T_C$, $N_0 = 0$, the heat capacity can be derived from equations (6) and (7) as

$$\begin{aligned} C_{T>T_C} &= \left(\frac{\partial E}{\partial T}\right)_N = \left(\frac{\partial E}{\partial T}\right)_{N,\mu} + \left(\frac{\partial E}{\partial \mu}\right)_{N,T}\left(\frac{\partial \mu}{\partial T}\right)_N \\ &= Nk_B\left\{\frac{\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta}}\left[\begin{matrix}[\eta(\eta-1)-2(\eta-1)j/\chi+j^2/\chi^2]K_{D'}(j/\chi)+(j/\chi)\\ \cdot(2\eta-3-2j/\chi)K_{D'+1}(j/\chi)+(j^2/\chi^2)K_{D'+2}(j/\chi)\end{matrix}\right]}{\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta-1}}K_{D'}(j/\chi)}\right. \\ &\quad \left. - \frac{\left\{\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta-1}}[(\eta-1-j/\chi)K_{D'}(j/\chi)+(j/\chi)K_{D'+1}(j/\chi)]\right\}^2}{\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta-1}}K_{D'}(j/\chi)\sum_{j=1}^{\infty}\frac{\exp(j\beta\mu)}{j^{D'+\eta-2}}K_{D'}(j/\chi)}\right\}, \end{aligned} \tag{13}$$

where the property of the modified Bessel function

$$x\frac{\mathrm{d}K_\nu(x)}{\mathrm{d}x} = \nu K_\nu(x) - xK_{\nu+1}(x) \tag{14}$$

is employed. When $T \leqslant T_C$, $\mu = mc^2$ and the heat capacity is found from equations (7) and (10) to be

$$C_{T\leqslant T_C} = \left(\frac{\partial E}{\partial T}\right)_N = Nk_B\left(\frac{T}{T_C}\right)^{D'+\eta-1} \times \frac{\sum_{j=1}^{\infty}\frac{\exp(j/\chi)}{j^{D'+\eta}}\left[\begin{array}{l}[\eta(\eta-1)-2(\eta-1)j/\chi+j^2/\chi^2]K_{D'}(j/\chi)+(j/\chi)\\ \cdot(2\eta-3-2j/\chi)K_{D'+1}(j/\chi)+(j^2/\chi^2)K_{D'+2}(j/\chi)\end{array}\right]}{\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}K_{D'}(j/\chi_C)}. \tag{15}$$

By using equations (13) and (15), the jump of the heat capacity between $T \to T_C^-$ and $T \to T_C^+$ is obtained as

$$\Delta C \equiv C_{T=T_C^-} - C_{T=T_C^+} = Nk_B\frac{\left\{\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}[(\eta-1-j/\chi_C)K_{D'}(j/\chi_C)+(j/\chi_C)K_{D'+1}(j/\chi_C)]\right\}^2}{\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}K_{D'}(j/\chi_C)\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+\eta-2}}K_{D'}(j/\chi_C)}. \tag{16}$$

3. Discussion

(1) Using equation (10), one can obtain one important condition that determines whether or not the trapped relativistic Bose system can condense at a nonzero temperature. For the system of massive particles, i.e., the parameter $\chi_C = k_BT_C/(mc^2)$ is finite, we have $j/\chi_C \gg 1$ for j greater than certain large value j_m. Thus, by using the expansion of the modified Bessel function for a large argument, i.e.,

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\exp(-x)\left(1+\frac{4\nu-1}{8x}+\frac{16\nu^4-40\nu^2+9}{128x^2}+\cdots\right), \tag{17}$$

equation (10) can be expressed as

$$N = \frac{\tilde{V}_C}{\lambda_C^D}\left(\frac{2}{\pi\chi_C}\right)^{1/2}\sum_{j=1}^{j_m-1}\frac{\exp(j/\chi_C)}{j^{D'+\eta-1}}K_{D'}(j/\chi_C) + \frac{\tilde{V}_C}{\lambda_C^D}\sum_{j=j_m}^{\infty}\frac{1}{j^{D'+\eta-1/2}}\left[1+\frac{4D'-1}{8}\left(\frac{\chi_C}{j}\right)+\frac{16D'^4-40D'^2+9}{128}\left(\frac{\chi_C}{j}\right)^2+\cdots\right]. \tag{18}$$

When $D'+\eta-1/2 > 1$, the sum $\sum_{j=j_m}^{\infty}$ in equation (18) is convergent, which implies that BEC can occur in the system. Considering $D' = (D+1)/2$ and $\eta = \sum_{k=1}^{D} 1/t_k$, we get a general condition for the BEC occurrence of a Bose gas with massive particles:

$$\sum_{k=1}^{D}\frac{1}{t_k}+\frac{D}{2} > 1. \tag{19}$$

It is interesting to note that this condition is the same as that of a nonrelativistic trapped Bose gas [12].

For the system of massless particles, i.e., $\chi_C = k_BT_C/(mc^2) \to \infty$, using the approximation $\exp(j/\chi_C) \approx 1$ and

$$K_\nu(x) \xrightarrow{x\to 0} \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, \tag{20}$$

one can get from equation (10) that

$$N = \frac{\tilde{V}_C}{\tilde{\lambda}_C^D}\zeta(\eta + D), \tag{21}$$

where

$$\tilde{\lambda}_C = \frac{hc}{2k_B T_C}\left[\frac{1}{\pi^{(D-1)/2}\Gamma(D/2+1/2)}\right]^{1/D} \tag{22}$$

is the ultrarelativistic thermal wavelength at $T = T_C$ [11], $\zeta(\nu) = \sum_{j=1}^{\infty} 1/j^{\nu}$ is the Riemann Zeta function. Since the Riemann Zeta function $\zeta(\nu)$ is convergent only when $\nu > 1$, the condition for BEC occurrence of a massless Bose gas is found from equation (21) to be

$$\sum_{k=1}^{D}\frac{1}{t_k} + D > 1. \tag{23}$$

It should be noted that equation (23) is valid only for a massless Bose gas with a conserved number of particles. For a phonon gas, because the number of phonons is not conserved, BEC does not take place.

Equations (19) and (23) indicate that the criteria for BEC occurrence relate not only to the dimensionality of space and characteristics of particles but also to the shape of the external potential. For example, in the case of $t_k \to \infty$, the external potential becomes

$$U(\boldsymbol{r}) = \begin{cases} 0 & |x_k| \leqslant L_k \\ \infty & |x_k| > L_k, \end{cases} \tag{24}$$

which corresponds to a D-dimensional box, and the criteria for the BEC occurrence are $D > 2$ and $D > 1$ for the systems of massive particles and massless particles, respectively. The results are just the same as those given in [2, 3]. In another case of $t_k = 2$, the external potential becomes a D-dimensional 'harmonic potential' and the criteria for BEC occurrence are $D > 1$ and $D > 2/3$ for the systems of massive and massless particles, respectively.

(2) Using equation (16) and the similar method mentioned above, we can obtain the general condition for the discontinuity of heat capacity at the critical temperature. For the system of massive particles, it is found that if

$$\sum_{k=1}^{D}\frac{1}{t_k} + \frac{D}{2} > 2, \tag{25}$$

there is a jump of the heat capacity at the critical temperature. Otherwise, there exists no jump. For the system of massless particles, however, the condition that the heat capacity is discontinuous at the critical temperature is found to be

$$\sum_{k=1}^{D}\frac{1}{t_k} + D > 2. \tag{26}$$

(3) It is seen from equation (10) that the critical temperature is dependent on the number of particles, the external potential and the rest mass of a particle. Let us introduce a parameter related to these quantities:

$$\chi_{C0} \equiv \frac{k_B T_{C0}}{mc^2} = \frac{1}{mc^2}\left[\frac{Nh^D}{\zeta(\eta + D/2)(2\pi m)^{D/2}}\prod_{i=1}^{D}\frac{\varepsilon_k^{1/t_k}}{(2L_k)\Gamma(1/t_k+1)}\right]^{1/(\eta+D/2)}, \tag{27}$$

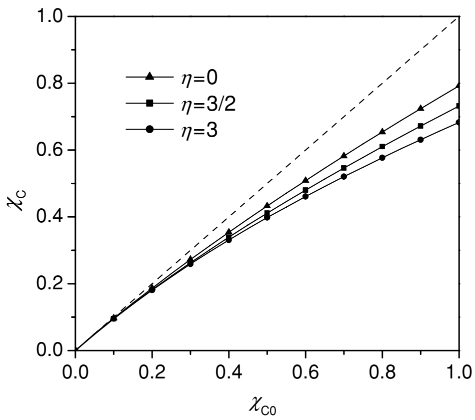

Figure 1. The scaled critical temperature $\chi_C = k_B T_C/(mc^2)$ as a function of the parameter χ_{C0}. The solid lines with symbols represent the results of the relativistic Bose gas for different η. The dashed line represents the result of the nonrelativistic approximation.

where T_{C0} is the critical temperature under the nonrelativistic limit [12]. By using equation (27), equation (10) can be expressed as

$$\chi_{C0}^{\eta+D/2} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\chi_C^{\eta+D/2-1/2}}{\zeta(\eta+D/2)} \sum_{j=1}^{\infty} \frac{\exp(j/\chi_C)}{j^{D'+\eta-1}} K_{D'}(j/\chi_C). \tag{28}$$

Figure 1 shows the scaled critical temperature $\chi_C = k_B T_C/(mc^2)$ as a function of the parameter χ_{C0} in the cases of $D = 3$ and different parameters η. The results are compared with the predictions based on the nonrelativistic approximation. It is shown that the relativistic effect results in the lowering of the critical temperature. For the system composed of particles of very small rest mass, which, according to equation (27), implies that the parameter χ_{C0} is large, the correction of the critical temperature due to the relativistic effect may be considerable.

(4) Using equations (6), (13) and (15), one can expound the dependence of the heat capacity on the temperature. Figure 2 shows the curves of the reduced heat capacity $C/(Nk_B)$ varying with the scaled temperature T/T_C, where (*a*) and (*b*) correspond to the cases of $D = 3$, $\eta = 0$ and $D = 3$, $\eta = 3/2$, respectively. It is found that the relativistic effect decreases the value of the heat capacity below a certain temperature but results in the increase in the heat capacity above this temperature (see the insets). The heat capacity is continuous in the case of $\eta = 0$ (figure 2(*a*)) but exists a gap at the critical temperature in the case of $\eta = 3/2$ (figure 2(*b*)). It can also be seen from figure 2(*b*) that the relativistic effect enlarges the gap of the heat capacity at the critical temperature.

(5) In the case of the nonrelativistic limit, i.e., $\chi = k_B T/(mc^2) \ll 1$ and $\chi_C = k_B T_C/(mc^2) \ll 1$, by using the approximation for the modified Bessel function:

$$K_\nu(x) \xrightarrow{x\gg 1} \sqrt{\frac{\pi}{2x}} \exp(-x). \tag{29}$$

Equations (10), (12), (13), (15) and (16) are, respectively, reduced to

$$T_C = \frac{1}{k_B} \left[\frac{Nh^D}{\zeta(\eta+D/2)(2\pi m)^{D/2}} \prod_{i=1}^{D} \frac{\varepsilon_k^{1/t_k}}{(2L_k)\Gamma(1/t_k+1)} \right]^{1/(\eta+D/2)}, \tag{30}$$

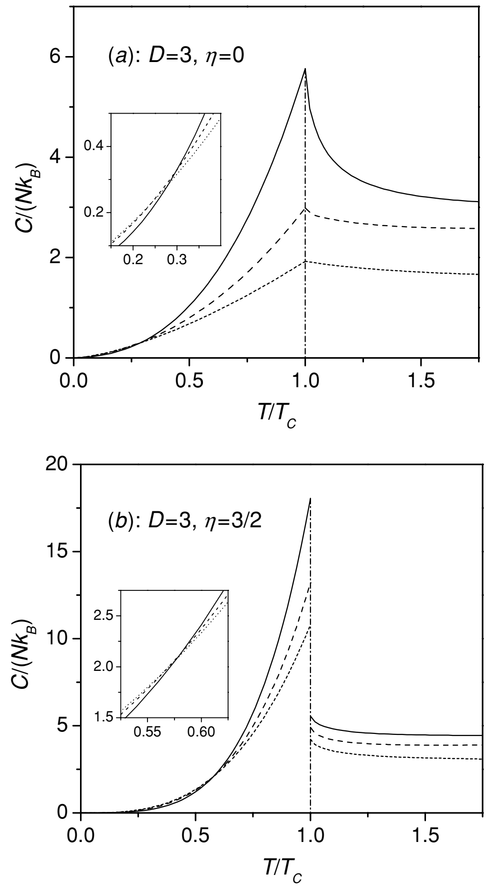

Figure 2. The heat capacity $C/(Nk_B)$ as a function of the scaled temperature T/T_C for the different values of χ_{C0}. The solid line and dashed lines represent the results of the relativistic Bose gas with $\chi_{C0} = 1.0$ and 0.2, respectively. The dot line represents the result of the nonrelativistic approximation.

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{\eta+D/2}, \tag{31}$$

$$C_{T>T_C} = Nk_B\left\{(\eta + D/2)(\eta + D/2 + 1)\frac{g_{\eta+D/2+1}(z)}{g_{\eta+D/2}(z)} - (\eta + D/2)^2\frac{g_{\eta+D/2}(z)}{g_{\eta+D/2-1}(z)}\right\}, \tag{32}$$

$$C_{T\leqslant T_C} = Nk_B(\eta + D/2)(\eta + D/2 + 1)\left(\frac{T}{T_C}\right)^{\eta+D/2}\frac{\zeta(\eta + D/2 + 1)}{\zeta(\eta + D/2)}, \tag{33}$$

$$\Delta C = Nk_B(\eta + D/2)^2\frac{\zeta(\eta + D/2)}{\zeta(\eta + D/2 - 1/2)}, \tag{34}$$

where $g_\nu(x) = \sum_{j=1}^{\infty} x^j/j^\nu$ is the expansion of the Bose integral and $z \equiv \exp[\beta(\mu - mc^2)]$, which, according to equations (6) and (29), is determined by

$$N = \frac{\tilde{V}}{\lambda^D}g_{\eta+D/2}(z) \qquad (T > T_C). \tag{35}$$

Equations (30)–(34) give, respectively, the expressions for the critical temperature, ground-state fraction, heat capacity at $T > T_C$ and $T \leqslant T_C$, and gap of the heat capacity at $T = T_C$ for a nonrelativistic ideal Bose gas trapped in a power-law potential. They are just the same as those obtained in [12] as long as $s = 2$ and $a = 1/(2m)$ in [12] are set. If it is further assumed that $t_k \to \infty$, i.e., $\eta = 0$, and $D = 3$, the properties of a nonrelativistic ideal Bose gas confined in a three-dimensional box, which have been discussed in many textbooks [15, 16], can be directly derived from the above results.

On the other hand, in the ultrarelativistic limit, i.e., $\chi = k_B T/(mc^2) \to \infty$ and $\chi_C = k_B T_C/(mc^2) \to \infty$, by using equation (20) and the approximations $\exp(j/\chi) \approx 1$ and $\exp(j/\chi_C) \approx 1$, the critical temperature, ground-state fraction, heat capacity at $T > T_C$ and $T \leqslant T_C$, and gap of the heat capacity at $T = T_C$ are found from equations (10), (12), (13), (15) and (16) to be

$$T_C = \frac{1}{k_B}\left[\frac{N(hc)^D}{2^D \pi^{(D-1)/2}\Gamma(D/2+1/2)\zeta(\eta+D)}\prod_{k=1}^{D}\frac{\varepsilon_k^{1/t_k}}{(2L_k)\Gamma(1/t_k+1)}\right]^{1/(\eta+D)}, \tag{36}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{\eta+D}, \tag{37}$$

$$C_{T>T_C} = Nk_B\left\{(\eta+D)(\eta+D+1)\frac{g_{\eta+D+1}(z)}{g_{\eta+D}(z)} - (\eta+D)^2\frac{g_{\eta+D}(z)}{g_{\eta+D-1}(z)}\right\}, \tag{38}$$

$$C_{T\leqslant T_C} = Nk_B(\eta+D)(\eta+D+1)\left(\frac{T}{T_C}\right)^{\eta+D}\frac{\zeta(\eta+D+1)}{\zeta(\eta+D)}, \tag{39}$$

$$\Delta C = Nk_B(\eta+D)^2\frac{\zeta(\eta+D)}{\zeta(\eta+D-1)}, \tag{40}$$

respectively, where $z \equiv \exp[\beta(\mu - mc^2)]$ is determined by

$$N = \frac{\tilde{V}}{\tilde{\lambda}^D} g_{\eta+D}(z) \qquad (T > T_C), \tag{41}$$

and

$$\tilde{\lambda} = \frac{hc}{2k_B T}\left[\frac{1}{\pi^{(D-1)/2}\Gamma(D/2+1/2)}\right]^{1/D} \tag{42}$$

is the ultrarelativistic thermal wavelength at temperature T [11]. Equations (36)–(40) are just the same as those obtained in [12] as long as $s = 1$ and $a = c$ in [12] are set. If $t_k \to \infty$ and $D = 3$ are further set, the properties of an ultrarelativistic Bose gas in a three-dimensional box can be obtained.

(6) Because of the general form of the external potential adopted, the expressions derived above are valid for a variety of Bose gases trapped in different external potentials corresponding to the different parameters t_k, ε_k and L_k.

If $t_k \to \infty$ is set, the expressions given above can be used to explore the BEC characteristics of a relativistic Bose gas confined in a D-dimensional box. For example, according to equations (10) and (11), the critical temperature is now determined by

$$N = \frac{V}{\lambda_C^D}\left(\frac{2}{\pi\chi_C}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'-1}}K_{D'}(j/\chi_C), \tag{43}$$

where $V = \prod_{k=1}^{D}(2L_k)$ is the D-dimensional volume of the box. Equation (43) is just the same as the results given in [2, 3] as long as $\lambda_C = \sqrt{h^2/(2\pi m k_B T_C)}$ and $\chi_C = k_B T_C/(mc^2)$

is substituted into it and the system of units such that $c = 1$, $k_B = 1$ and $\hbar = h/(2\pi) = 1$ is employed. In the particular case of $D = 3$, equation (43) is simplified as

$$N = \frac{V}{\lambda_C^3}\left(\frac{2}{\pi\chi_C}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j}K_2(j/\chi_C). \tag{44}$$

It is found that equation (44) is the same as the result obtained in [1] if the necessary mathematical transformation is done.

If $t_k = 2$ and $\varepsilon_k\big/L_k^{t_k} = \gamma_k/2$ (γ_k is a positive constant) are chosen, the above expressions will give the properties of a Bose gas trapped in a D-dimensional 'harmonic potential', which have been widely investigated under the nonrelativistic limit [13, 14]. For example, it is found from equations (10) and (11) that, in the case of 'harmonic trap', the critical temperature is determined by

$$N = \left(\frac{k_B T_C}{\hbar\varpi}\right)^D\left(\frac{2}{\pi\chi_C}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\exp(j/\chi_C)}{j^{D'+1/2}}K_{D'}(j/\chi_C), \tag{45}$$

where $\varpi \equiv \left[\prod_{k=1}^{D}\gamma_k^{1/2}\right]^{1/D}\big/m^{1/2}$. In the case of nonrelativistic limit, equation (45) is reduced to

$$T_C = \frac{\hbar\varpi}{k_B}\left[\frac{N}{\zeta(D)}\right]^{1/D}, \tag{46}$$

which is the familiar expression for the critical temperature of a nonrelativistic ideal Bose gas trapped in a harmonic potential [13].

4. Conclusions

In summary, we have studied the Bose–Einstein condensation of a relativistic Bose gas trapped in a D-dimensional generic power-law potential. Some important parameters are derived and the general criteria on BEC occurrence and the discontinuity of the heat capacity at the critical temperature are given. It is shown that the criteria on the BEC occurrence and the discontinuity of the heat capacity at the critical temperature are different for the systems with massive and massless particles. The relativistic effect lowers the critical temperature of BEC but enlarges the gap of the heat capacity at the critical temperature.

Although one relativistic Bose system trapped in an external potential is studied only, the results derived in the present paper can be used to explore the BEC properties of a variety of Bose systems in a unified way. Therefore, the results obtained here are more general and useful.

It should be noted here that the present paper mainly concentrates on discussing the corrections to the BEC properties resulted from the relativistic effect and does not consider the possibility of the particle–antiparticle pair production. It is shown that taking the pair production into account will lead to several novel characteristics about a relativistic Bose system. This will be discussed in the proceeding work.

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